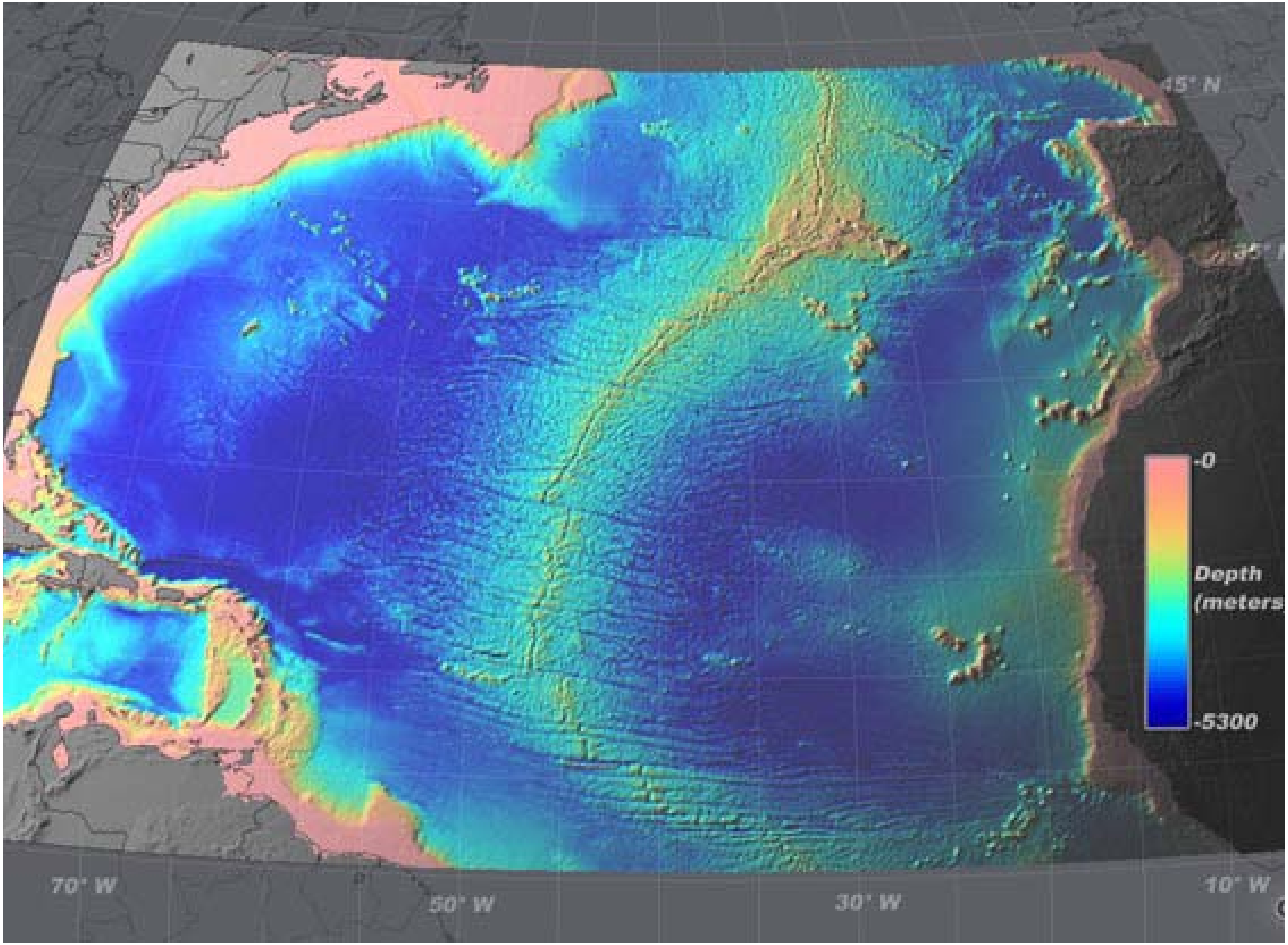
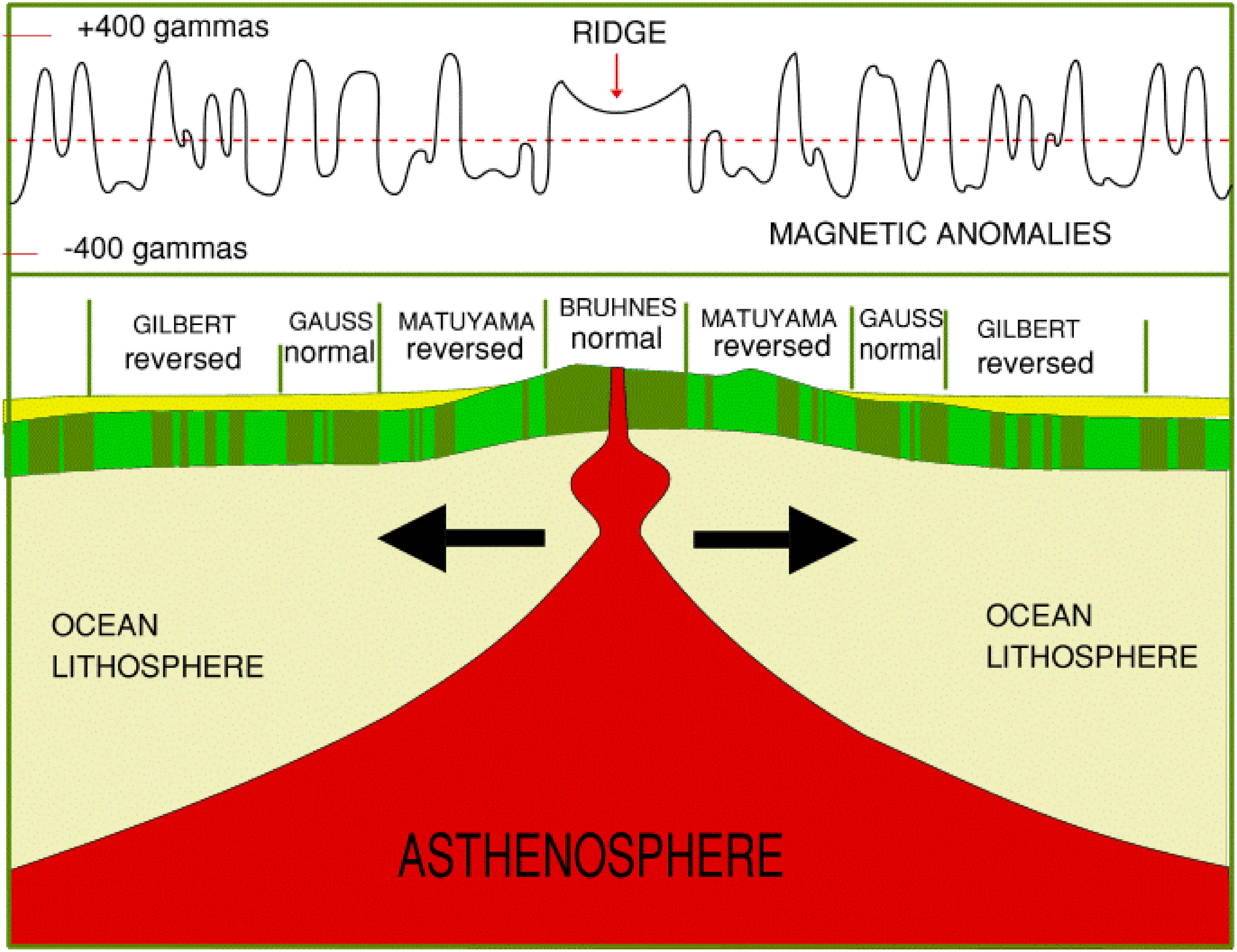
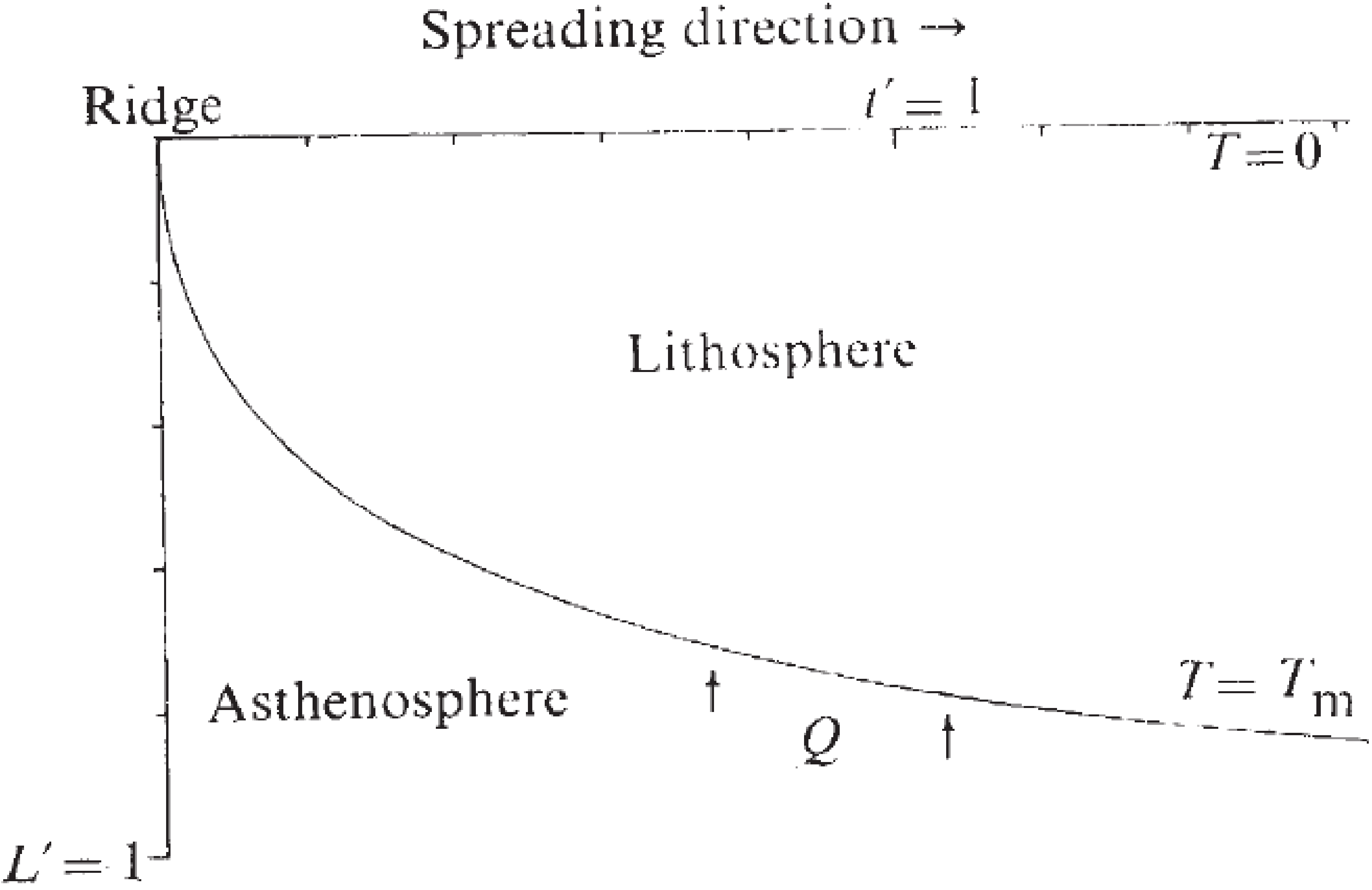
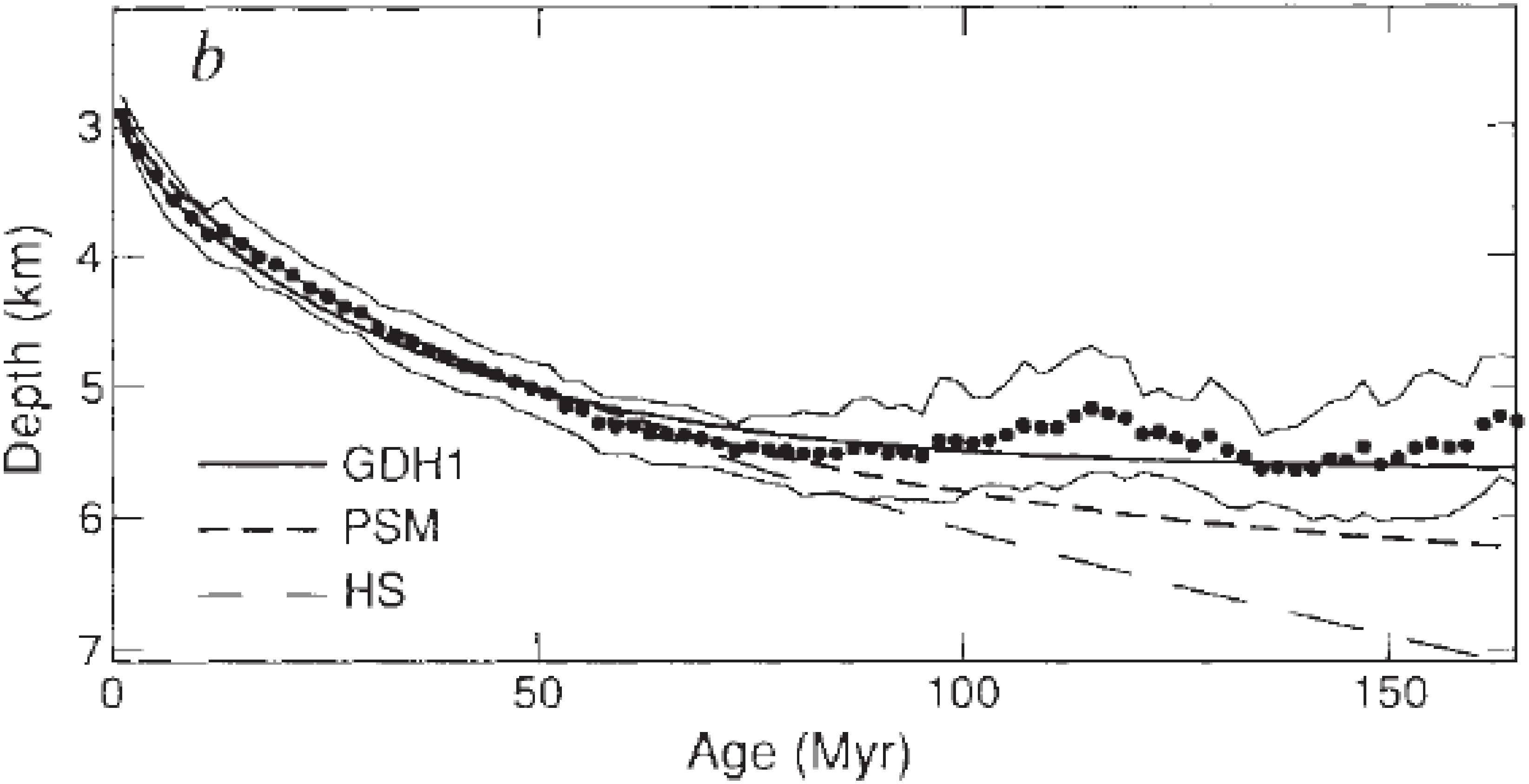
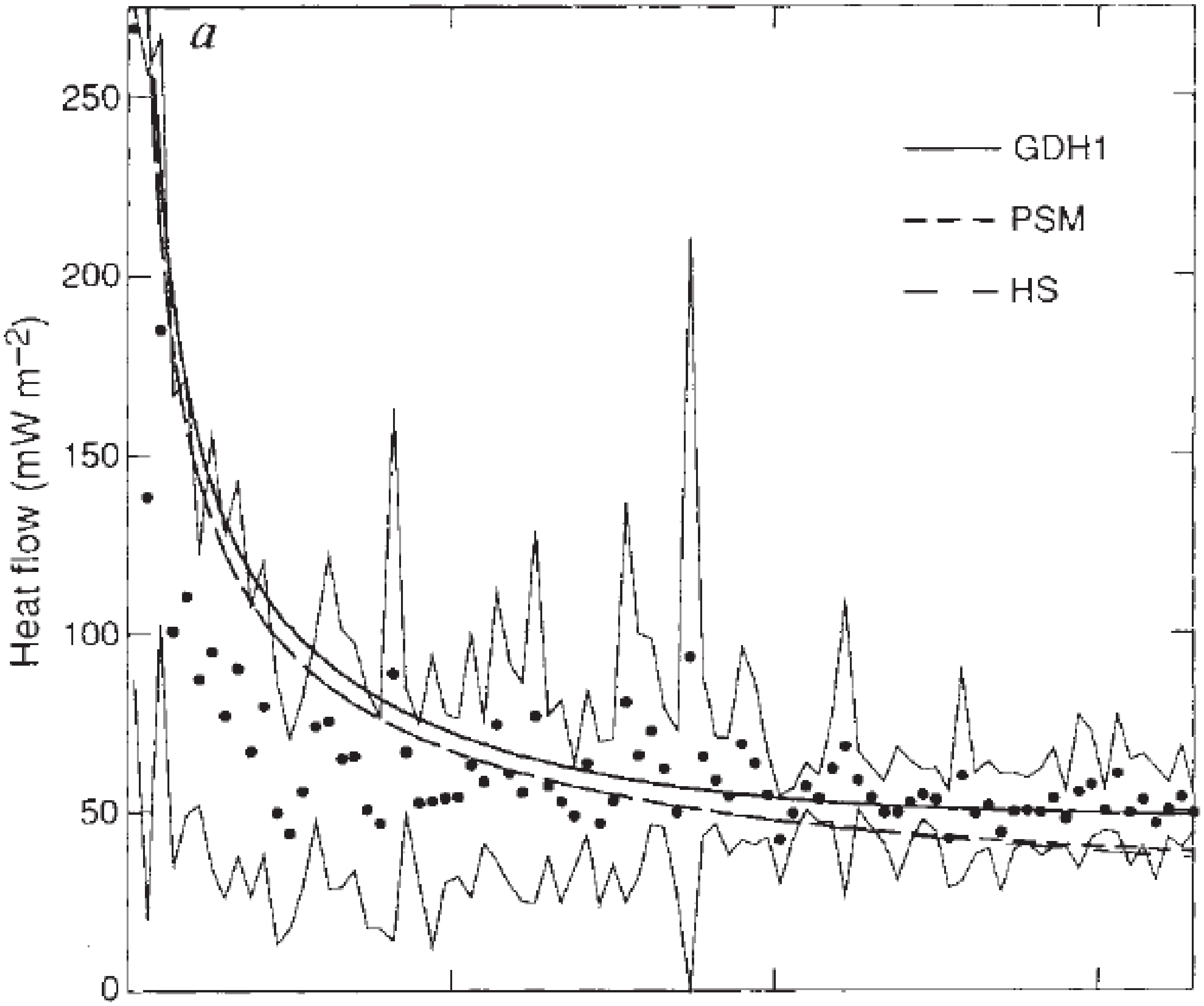

45° N
35° N
-0
Depth
(meters)
-5300
70° W
50° W
30° W
10° W


+400 gammas
RIDGE
-400 gammas
MAGNETIC ANOMALIES
GILBERT reversed
GAUSS normal
MATUYAMA reversed
BRUHNES normal
MATUYAMA reversed
GAUSS normal
GILBERT reversed
OCEAN LITHOSPHERE
OCEAN LITHOSPHERE
ASTHENOSPHERE


Spreading direction →
Ridge
$t' = 1$
$T = 0$
Lithosphere
Asthenosphere
$T = T_m$
Q
$L' = 1$


b
Depth (km)
3
4
5
6
7
GDH1
PSM
HS
0
50
100
150
Age (Myr)


a
Heat flow (mW m⁻²)
0
50
100
150
200
250
GDH1
PSM
HS

Heat equation: $\partial_t T = \kappa \, \partial_{xx} T$

Put: $u(x,t) = T(x,t) - T_0$

Require: $u(x,0) = 0$

$u(0,t) = T_{\mathrm{surf}}(t) - T_0$

$\lim_{x \to \infty} u(x,t) < \infty$

So: $\partial_t u = \kappa \, \partial_{xx} u$

Laplace transform $t \to s$:

$$\frac{d^2}{dx^2} U(x, s) = \frac{1}{\kappa} s \, U(x, s)$$

Solution:

$$U(x, s) = U(0, s) \exp\left(-x \sqrt{\frac{s}{\kappa}}\right)$$

Lévy kernel:

$$f_{\text{levy}}(t; \mu) = \sqrt{\frac{\mu}{2\pi}} \, t^{-3/2} \exp\left(-\frac{\mu}{2t}\right)$$

$$F_{\text{levy}}(s; \mu) = \exp\left(-\sqrt{2 \mu s}\right)$$

Thus, solution is

$$U(x, s) = U(0, s) \, F_{\text{levy}}\left(s; \frac{x^2}{2\kappa}\right)$$

This is a convolution in the time domain:

$$u(x,t) = u(0,t) \otimes f_{\text{levy}}\left(t; \frac{x^2}{2\kappa}\right)$$

so the full solution is:

$$T(x,t) = T_0 + T(0,t) \otimes \left\{ \frac{x}{2\sqrt{\pi\kappa}}\, t^{-3/2} \exp\left(-\frac{x^2}{4\pi\kappa}\right) \right\}$$

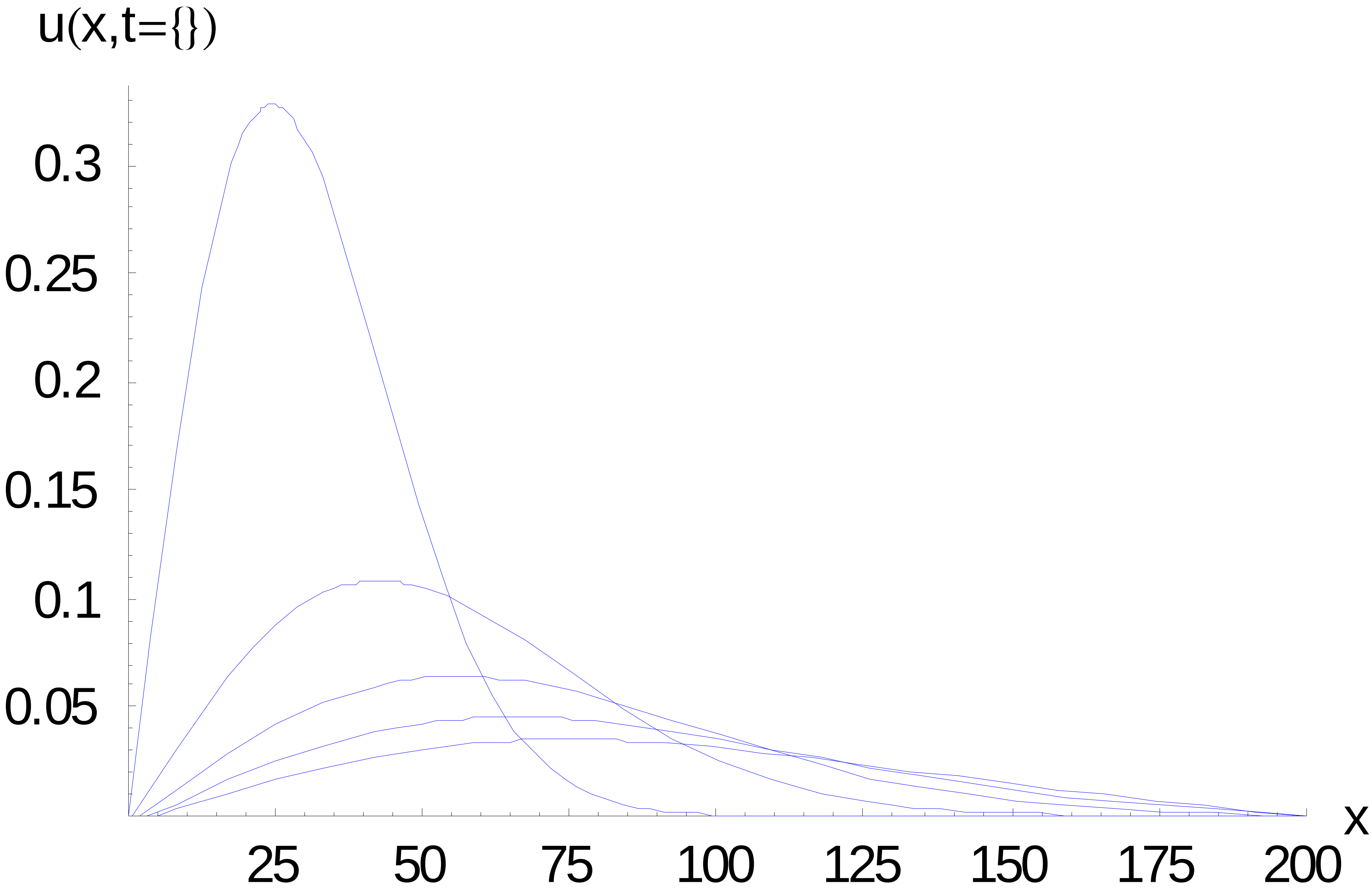
For flash heating at the boundary:

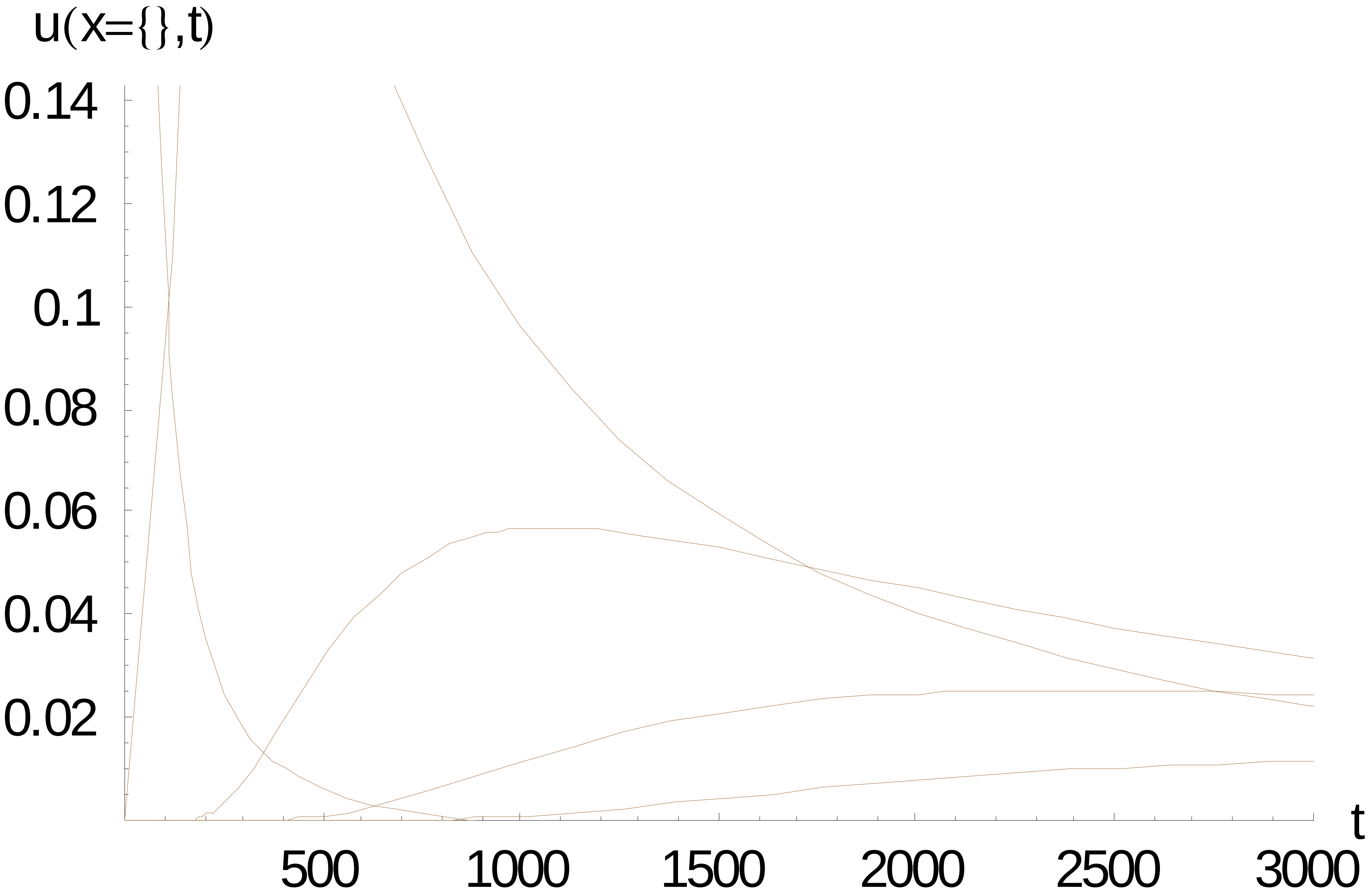
$$T_0 = 0 \quad , \quad T(x,t) = \delta(t)$$

thus:

$$T(x,t) = \frac{x}{2\sqrt{\pi\kappa}}\, t^{-3/2} \exp\left(-\frac{x^2}{4\pi\kappa}\right)$$

which is the ***Green's function*** for half-space cooling.


u(x,t={})
0.3
0.25
0.2
0.15
0.1
0.05
x
25
50
75
100
125
150
175
200



u(x={},t)
0.14
0.12
0.1
0.08
0.06
0.04
0.02
500
1000
1500
2000
2500
3000
t

Heat flux as Laplace transform:

$$\Phi(x,s) = -\kappa\,\partial_x U(x,s) = -\kappa\,U(0,s)\sqrt{\frac{s}{\kappa}}\;\exp\left(-x\sqrt{\frac{s}{\kappa}}\right)$$

so:

$$\Phi(x,s) = \sqrt{\kappa}\;\sqrt{s}\;U(x,s)$$

Fractional derivative in Riemann-Liouville form:

$${}_0D_t^p = \frac{1}{\Gamma(n-p)}\,\frac{d^n}{dt^n}\left\{\int_0^t \frac{f(\tau)}{(t-\tau)^{p-n+1}}\,d\tau\right\}$$

Laplace transform of fractional derivative operator, $t \rightarrow s$:

$$\mathfrak{L}\{{}_0D_t^p;\mathrm{s}\} = s^p F(s) - \sum_{k=0}^{n-1} s^k\left[{}_0D_t^{p-k-1} f(t)\right]_{t=0}$$

For $0 \leq p < 1$:

$$\mathfrak{L}\{{}_0D_t^p;\mathrm{s}\} = s^p F(s) - \left[{}_0D_t^{p-1} f(t)\right]_{t=0}$$

At boundary in thermal problem:

$$\left[{}_0D_t^{-1/2}\,u(0,t)\right]_{t=0} = 0$$

if *u(0,t)* is sufficiently (*p-1=-1/2<0)* differentiable

Thus the Laplace transformed heat flux is:

$$\Phi(x,s) = \sqrt{\kappa}\;\mathfrak{L}\left\{{}_0D_t^{1/2}\,u(x,t);s\right\}$$

which in the time-domain is:

$$\boxed{\phi(x,t) = \sqrt{\kappa}\;{}_0D_t^{1/2}\,u(x,t)}$$

Laplace transformed heat equation:

$$\frac{d^2}{dx^2}\, U(x, s) = \frac{1}{\kappa}\, s\, U(x, s)$$

Generic solution:

$$U(x, s) = A(s) \exp\left(x \sqrt{\frac{s}{\kappa}}\right) + B(s) \exp\left(-x \sqrt{\frac{s}{\kappa}}\right)$$

Boundary condition: $u(0, t) = u_0\, \delta(t)$

so: $A + B = U_0$

Laplace-transformed flux:

$$\Phi(x, s) = -\kappa \frac{d}{dx}\, U(x, s)$$

$$= -\sqrt{\kappa\, s}\, \left\{A \exp\left(x \sqrt{\frac{s}{\kappa}}\right) - B \exp\left(-x \sqrt{\frac{s}{\kappa}}\right)\right\}$$

Zero RHS flux at $x=L$:

$$\phi(L,t) = 0 \iff \frac{dU}{dx}\Big|_{x=L} = 0$$

So:

$$A\exp\left(L\sqrt{\frac{s}{\kappa}}\right) = B\exp\left(-L\sqrt{\frac{s}{\kappa}}\right)$$

Or:

$$A = B\exp\left(-2L\sqrt{\frac{s}{\kappa}}\right)$$

Which gives:

$$B(s) = U_0\left[1+\exp\left(-2L\sqrt{\frac{s}{\kappa}}\right)\right]$$

Substituting:

$$U(x,s) = U_0\left(\frac{\exp\left(-2L\sqrt{\frac{s}{\kappa}}\right)\exp\left(x\sqrt{\frac{s}{\kappa}}\right)+\exp\left(-x\sqrt{\frac{s}{\kappa}}\right)}{1+\exp\left(-2L\sqrt{\frac{s}{\kappa}}\right)}\right)$$

or: $$U(x,s) = U_0\left(\frac{\exp\left((x-L)\sqrt{\frac{s}{\kappa}}\right)+\exp\left(-(x-L)\sqrt{\frac{s}{\kappa}}\right)}{\exp\left(L\sqrt{\frac{s}{\kappa}}\right)+\exp\left(-L\sqrt{\frac{s}{\kappa}}\right)}\right)$$

... how can this be inverse Laplace-transformed?

Note that at the limit:

$$\lim_{L\to\infty}U(L,s)| = U_0\ \exp\left(-x\sqrt{\frac{s}{\kappa}}\right) = U_0\ F_{\mathrm{levy}}\left(s;\frac{x^2}{2\,\kappa}\right)$$