Extreme Value Models for Processes with Heavy-Tailed Interarrivals

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Goals of Extreme Value Theory

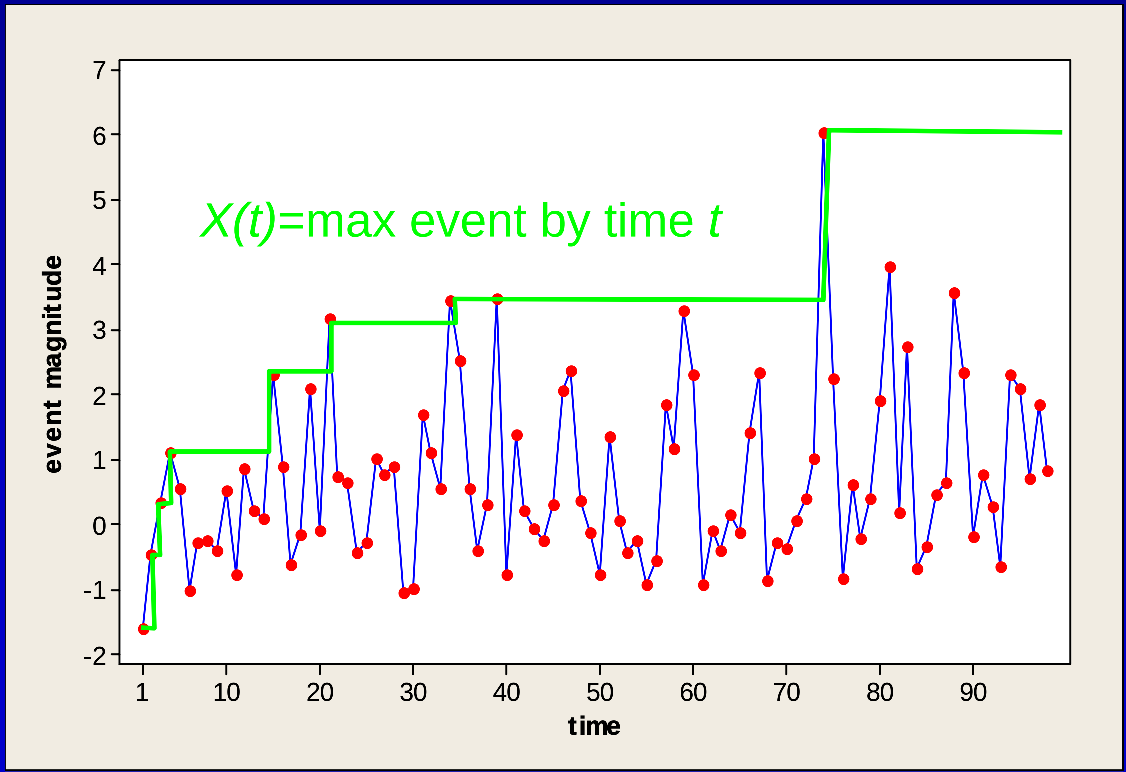
- Make predictions about the recurrence of highly unusual events

➢Flood

➢Earthquake

➢Stock price fluctuation

➢Solar Flare



Classical EV Models

Stochastic models in which the magnitude of extreme events is random

For iid random variables $X_1,\ldots,X_m$, compute the distribution function of the maxima by

$$P\left\{\max\ X_i \le x\right\} = P\left\{X_1 \le x, \ldots, X_m \le x\right\} = F^m\left(x\right)$$

Extreme Value Distributions
limiting max-stable distributions

Type I: Gumbel
Distribution of event magnitudes has thin tails

Type II: Frechet
Distribution of event magnitudes has heavy tails

Stochastic Model Underlying Classical EV Theory

- Poisson Process
  - iid events of arbitrary distribution are separated by exponential or non-random wait times
  - the number of events by time *t* is Poisson

$$N(t) \sim Poisson(\lambda t) \quad 1/\lambda = E(\text{exceedances /yr})$$

Estimation of a T-year event for a partial duration series

$X_i =$ peak magnitudes $\sim F(x)$

x = arbitrary threshold

$$N_x(t) \sim Poisson\left(\lambda_x t\right) \quad 1/\lambda_x = E\left(\text{exceedances of threshold x/yr}\right)$$

$$P\left\{N_x(t) = n\right\} = \frac{\left(\lambda_x t\right)^n}{n!} e^{-\lambda_x t},\ n = 0,1,2...$$

$$\lambda_x = \lambda\left[P\left(X > x\right)\right] = \lambda\left[1 - F(x)\right]$$

Annual max contribution:

$$P\left\{N_x(1) = 0\right\} = \frac{\left(\lambda_x t\right)^0}{0!} e^{-\lambda_x t} = e^{-\lambda[1-F(x)]}$$

Estimation of a T-year event for a partial duration series

Annual max contribution:

$$P\{N_x(1)=0\} = \frac{(\lambda_x t)^0}{0!} e^{-\lambda_x t} = e^{-\lambda[1-F(x)]}$$

Event magnitude distribution	Distribution of annual maxima
Exponential →	Gumbel
Pareto →	Frechet

Extreme Value (EV) Theory

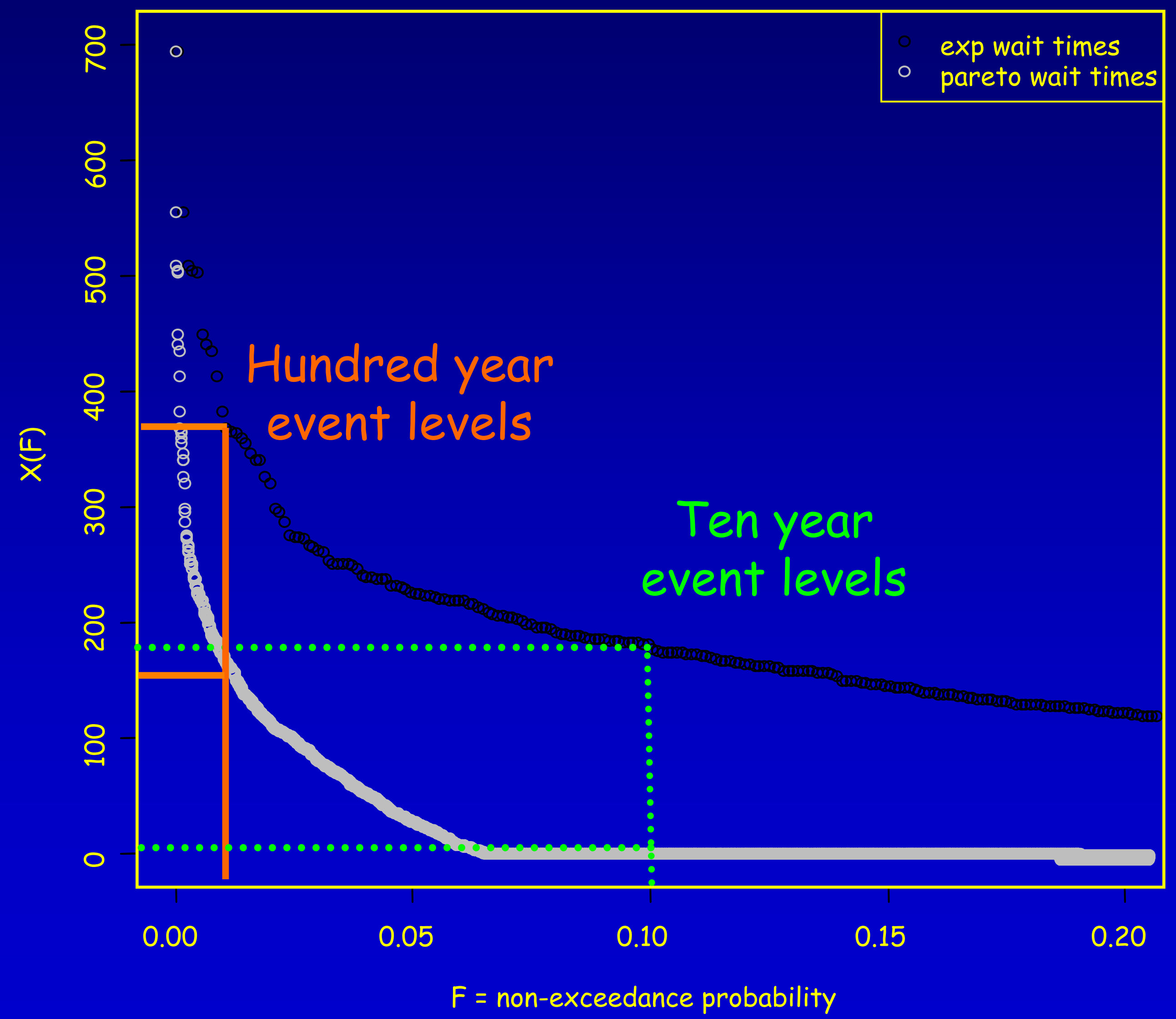
	thin-tailed interarrivals	heavy-tailed interarrivals
thin-tailed event magnitudes	Gumbel	?
heavy-tailed event magnitudes	Fréchet	?

Heavy-tailed Extreme Value Theory

- Power law interarrivals between extreme events have been observed in
  - storm origins
  - raindrop arrival on the ground
  - alluvial events
  - earthquakes
  - solar flares

- EV models are not applicable in these cases

Why are heavy-tailed interarrivals significant?



Continuous Time Random Walk (CTRW)

$$X(t) = S(N(t)) = \sum_{n=1}^{N(t)} \vec{Y}_n$$

Total distance traveled is the sum of each jump length

$$S(n) = \vec{Y}_1 + \vec{Y}_2 + \ldots + \vec{Y}_n$$

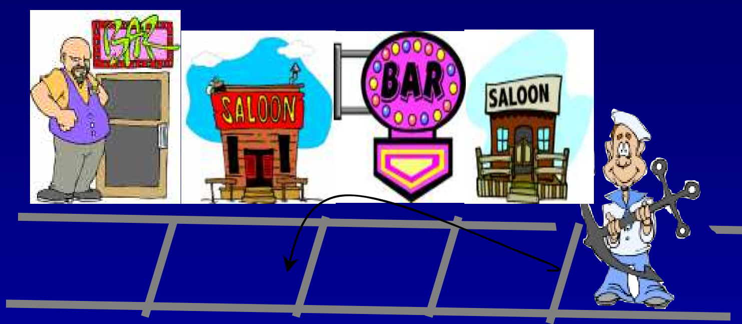
There can be pauses between jumps

$$T(n) = J_1 + \ldots + J_n$$

$$N(t) = \max\{n : T(n) \le t\}$$

Jump lengths and pause times are random and each follows a probability distribution

Continuous Time Random Max (CTRM)



$$X(t) = M_{N_t} = \max\left(Y_1, Y_2, \ldots Y_{N_t}\right)$$

What is the largest extreme event by random time *t?*

There can be heavy tailed pauses between extreme events

$$M_{N_t} = \max\left(Y_1, Y_2, \ldots Y_{N_t}\right)$$

$$T(n) = J_1 + \ldots + J_n$$

$$N(t) = \max\left\{n : T(n) \le t\right\}$$

Event magnitude and inter-arrivals are random and each follows a probability distribution

CTRM and CTRW

The relationship between the counting process and the interarrival time process $\{N_t > n\} = \{T_n \le t\}$

can be used to determine the density of the counting process.

$$P\{M(t) \le x\} = \frac{t}{\gamma} \int_0^\infty G(x)^s \, s^{-1-1/\gamma} g_\gamma\left(ts^{-1/\gamma}\right) ds$$

Weibull, Frechet

Stable Subordinator

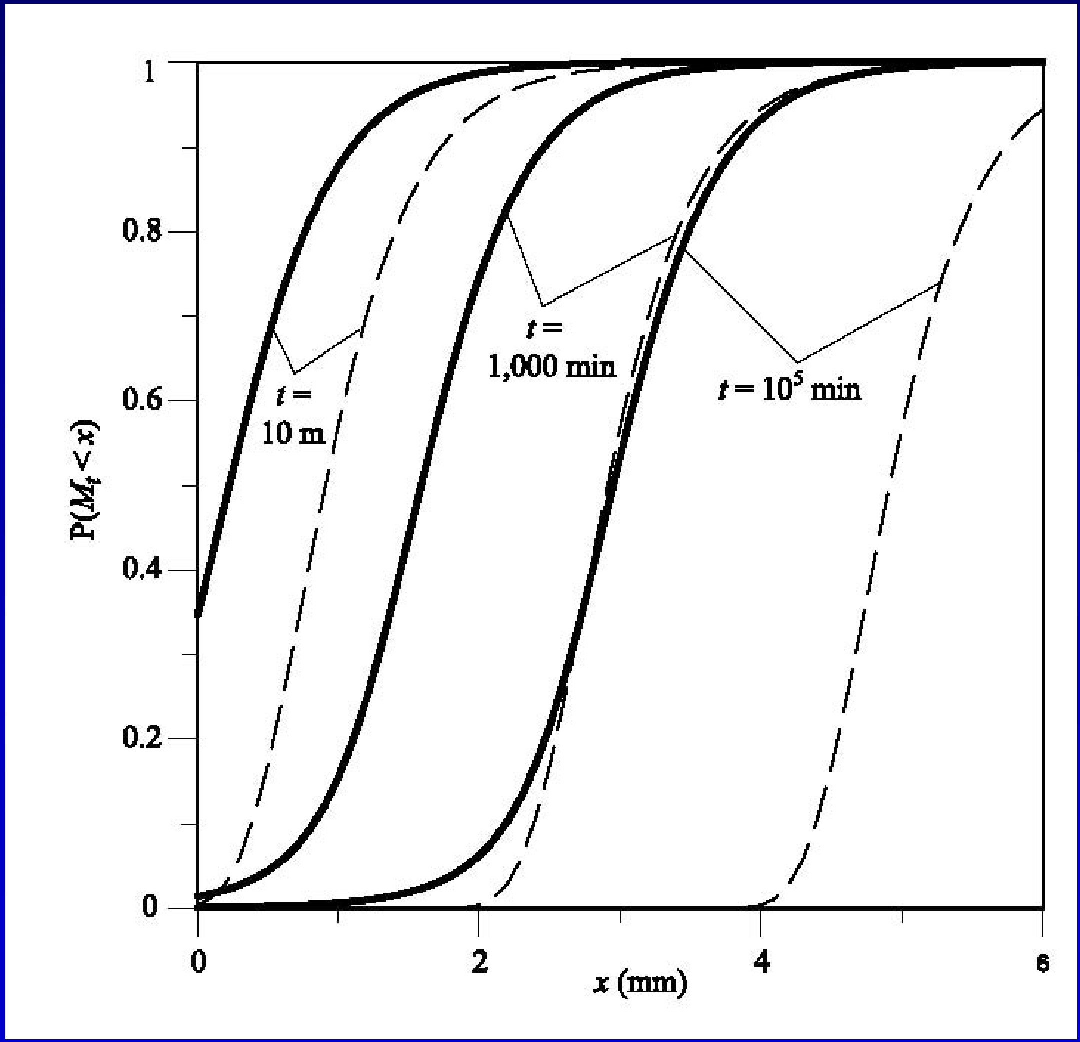
Results –
General Extreme Value distributions

	thin-tailed interarrivals	heavy-tailed interarrivals
thin-tailed event magnitudes	Gumbel	Subordinated Gumbel
heavy-tailed event magnitudes	Fréchet	Subordinated Fréchet

Example: Raindrop arrival on the ground

(Lavergnat, J. and P. Gole, 2006. J. Hydrol., 329 (1-2), 8-19)

CDF for maximum drop size at 10, 1,000, and 100,000 seconds for a Poisson arrival process versus a power-law interarrival process



Summary

- Classical EV theory assumes exponential or non-random interarrivals between extreme events or exceedances
- A continuous time random max process is like a CTRW with sums replaced by maxima
- CTRM accommodate arbitrary interarrival distributions
- General extreme value distributions are obtained by subordinating classical distributions